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An LQP method for pseudomonotone variational inequalities

Abdellah Bnouhachem

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Abstract In this paper, we proposed a modified Logarithmic-Quadratic Proximal (LQP) method [Auslender et al.: Comput. Optim. Appl. **12**, 31–40 (1999)] for solving variational inequalities problems. We solved the problem approximately, with constructive accuracy criterion. We show that the method is globally convergence under that the operator is pseudomonotone which is weaker than the monotonicity and the solution set is nonempty. Some preliminary computational results are given.

Keywords Variational inequality · Pseudomonotone operator · Logarithmic-quadratic proximal method

1. Introduction

A classical variational inequality problem, denoted by VI(F, Ω), is to find a vector $x^* \in \Omega$ such that

$$F(x^*)^T(y - x^*) \geq 0, \quad \forall y \in \Omega, \tag{1}$$

where $\Omega \subset R^n$ is a nonempty closed convex subset of R^n and F is a continuous mapping from R^n into itself. VI(F, Ω) includes nonlinear complementarity problems (when $\Omega = R^n_+$) and system of nonlinear equations (when $\Omega = R^n$). The present analysis mainly focused on the case where $\Omega = R^n_+$.

Variational inequality problems have many important applications in economics, operations research and nonlinear analysis and have been studied by many researchers [3, 5, 6, 8, 9, 12].

It is well known that the VI(F, R^n_+) problem can be alternatively formulated as finding the zero point of the operator $T(x) = F(x) + N_{R^n_+}(x)$, i.e., find $x^* \in R^n_+$ such that $0 \in T(x^*)$, where $N_{R^n_+}(\cdot)$ is the normal cone operator to R^n_+ defined by

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A. Bnouhachem (✉)
School of Management Science and Engineering, Nanjing University, Nanjing, 210093, P. R. China
e-mail: babedallah@yahoo.com

$$N_{R^n_+}(x) = \begin{cases} \{y \in R^n : y^T(v - x) \le 0, \quad \forall v \in R^n_+\} & \text{if} \quad x \in R^n_+, \\ \emptyset & \text{otherwise.} \end{cases}$$

A classical method to solve this problem is the proximal point algorithm, which starting with any vector $x^0 \in R^n_+$ and $\beta_k \ge \beta > 0$, iteratively updates x^{k+1} conforming the following problem:

$$0 \in \beta_k T(x) + \nabla_x q(x, x^k), \tag{2}$$

where

$$q(x, x^k) = \tfrac{1}{2}\|x - x^k\|^2, \tag{3}$$

is a quadratic function of x. Motivation for studying the algorithms of problem (2) could be found in several studies [7, 13, 15], in place of usual quadratic term where many reseachers have used some nonlinear functions $r(x, x^k)$. For instance, we quoted reference [4] for the iterative schemes of the form (2) using the bregman-based functional instead of (3).

Recently, Auslender et al. [2] have proposed a new type of proximal interior method through replacing the quadratic function (3) by $d_\phi(x, x^k)$ which could be defined as

$$d_\phi(x, y) = \sum_{j=1}^{n} y_j^2 \phi(y_j^{-1} x_j).$$

The fundamental difference here is that the term d_ϕ is used to force the iterates $\{x^{k+1}\}$ to stay in the interior of the nonegative orthant R^n_{++}.

Among the possible choices of ϕ, there exists a particular one which enjoys several attractive properties for developping efficient algorithms to solve VI(F, R^n_+).

Let $\nu > \mu > 0$ be given fixed parameters, and define

$$\phi(t) = \begin{cases} \frac{\nu}{2}(t-1)^2 + \mu(t - \log t - 1) & \text{if} \quad t > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

In [1], Auslender et al. used a very special logarithmic-quadratic proximal (LQP) method (with $\nu = 2, \mu = 1$) for solving variational inequalities over polyhedra.

Let $\mu \in (0, 1)$ be a constant, in this paper we consider another function ϕ defined by

$$\phi(t) = \begin{cases} \frac{1}{2}(t-1)^2 + \mu(t \log t - t + 1) & \text{if} \quad t > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Then the problem (2) becomes for given $x^k \in R^n_+$ and $\beta_k \ge \beta > 0$, the new iterate x^{k+1} is unique solution of the following set-valued equation:

$$0 \in \beta_k T(x) + \nabla_x Q(x, x^k), \tag{4}$$

where

$$Q(x, x^k) = \begin{cases} \frac{1}{2}\|x - x^k\|^2 + \mu \sum_{j=1}^{n} (x_j^k)^2 \left(\frac{x_j}{x_j^k} \log \frac{x_j}{x_j^k} - \frac{x_j}{x_j^k} + 1 \right), & \text{if} \quad x \in R^n_{++}; \\ +\infty, & \text{otherwise.} \end{cases} \tag{5}$$

It is easy to see that

$$\nabla_x Q(x, x^k) = x - x^k + \mu X_k \log \frac{x}{x^k},$$

where $X_k = diag(x_1^k, \ldots, x_n^k)$ and $\log \frac{x}{x^k} = (\log \frac{x_1}{x_1^k}, \ldots, \log \frac{x_n}{x_n^k})^T$.

Then the problem (4)–(5) is equivalent to the following systems of nonlinear equations

$$\beta_k F(x) + x - x^k + \mu X_k \log \frac{x}{x^k} = 0. \tag{6}$$

It is more practical to find approximate solutions of (6) rather than the exact solutions due the fact that in general it exclude some practical applications. Driven by the fact of eliminating this drawback, in this paper, we presented a prediction–correction method with more relaxed conditions than [1] to solve (6) approximately.

The remaining part of the paper is structured as follows. In the next Section 2 we summarize some basic properties used in this paper. In Section 3, we have presented and analyzed our method followed by Section 4, where its global convergence is proved. Section 5 deals with some preliminary results of the proposed method.

2. Preliminaries

We list some important results which will be required in our following analysis.

First, we denote $P_{R^n_+}(\cdot)$ as the projection under the Euclidean norm, i.e.,

$$P_{R^n_+}(z) = \min\{\|z - x\| \mid x \in R^n_+\}.$$

From the above definition, it follows that

$$(v - P_{R^n_+}(v))^T (u - P_{R^n_+}(v)) \le 0, \quad \forall u \in R^n_+, \quad \forall v \in R^n. \tag{7}$$

From (7), it is easy to verify that

$$\|P_{R^n_+}(v) - P_{R^n_+}(u)\| \le \|v - u\|, \quad \forall u, v \in R^n, \tag{8}$$

and

$$\|P_{R^n_+}(v) - u\|^2 \le \|v - u\|^2 - \|v - P_{R^n_+}(v)\|^2, \quad \forall v \in R^n, u \in R^n_+. \tag{9}$$

Definition 2.1 $\forall u, v \in R^n$, the operator $F\colon R^n \to R^n$ is said to be pseudomonotone, if

$$(v - u)^T F(u) \ge 0 \Rightarrow (v - u)^T F(v) \ge 0.$$

The following Lemma plays a crucial role in the analysis via Lemma 3.1.

Lemma 2.1 *For given $x^k > 0$ and $q \in R^n$, let x be the positive solution of the following equation:*

$$q + x - x^k + \mu X_k \log \frac{x}{x^k} = 0, \tag{10}$$

where $X_k = \mathrm{diag}(x^k_1, \ldots, x^k_n)$ and $\log \frac{x}{x^k} = (\log \frac{x_1}{x^k_1}, \ldots, \log \frac{x_n}{x^k_n})$, then for any $y \ge 0$ we have

$$(y - x)^T q \ge \tfrac{1+\mu}{2}\left(\|x - y\|^2 - \|x^k - y\|^2\right) + \tfrac{1-\mu}{2}\|x^k - x\|^2. \tag{11}$$

Proof For each $t > 0$ we have $1 - \frac{1}{t} \le \log t \le t - 1$, then we obtain after multiplication by $y_j x^k_j \ge 0$ for each $j = 1, \ldots, n$,

$$y_j x^k_j \log \frac{x_j}{x^j_k} \le y_j x^k_j \left(\frac{x_j}{x^k_j} - 1 \right) = y_j (x_j - x^k_j),$$

and after multiplication by $x_j x_j^k \geq 0$ for each $j = 1, \ldots, n$,

$$-x_j x_j^k \log \frac{x_j}{x_j^k} \leq x_j x_j^k \left(\frac{x_j^k}{x_j} - 1 \right) = x_j^k (x_j^k - x_j),$$

adding the two inequalities, then obtained

$$(y_j - x_j)(x_j - x_j^k + \mu x_j^k \log \frac{x_j}{x_j^k}) \leq \mu (y_j - x_j^k)(x_j - x_j^k) + (x_j - x_j^k)(y_j - x_j).$$

Using the identities

$$(y_j - x_j^k)(x_j - x_j^k) = \tfrac{1}{2} \left((x_j - x_j^k)^2 - (x_j - y_j)^2 + (y_j - x_j^k)^2 \right),$$
$$(x_j - x_j^k)(y_j - x_j) = \tfrac{1}{2} \left((y_j - x_j^k)^2 - (y_j - x_j)^2 - (x_j - x_j^k)^2 \right),$$

and recalling (10) thus obtained

$$(x_j - y_j)(-q_j) \geq \frac{1+\mu}{2} \left((x_j - y_j)^2 - (x_j^k - y_j)^2 \right) + \frac{1-\mu}{2} (x_j^k - x_j)^2.$$

Summing over $j = 1, \ldots, n$, encountered (11). □

In course we always assume that the function F is pseudomonotone and the solution set of VI(F, R_+^n), denoted by Ω^*, is nonempty.

3. The proposed method

At the kth iteration, LQP method finds the exact solution for the following system of equations:

$$\beta_k F(x) + x - x^k + \mu X_k \log \frac{x}{x^k} = 0. \tag{12}$$

We now present an LQP method-based prediction–correction method for solving VI(F, R_+^n). For given $x^k > 0$ and $\beta_k > 0$, each iteration of the proposed method consists of two steps, the first step offers a predictor $\tilde{x}^k$ and the second step produces the new iterate x^{k+1}.

Prediction step: Find an approximate solution $\tilde{x}^k$ of (12), called predictor, such that

$$0 \approx \beta_k F(\tilde{x}^k) + \tilde{x}^k - x^k + \mu X_k \log \frac{\tilde{x}^k}{x^k} = \xi^k, \tag{13}$$

and ξ^k which satisfies

$$\|\xi^k\| \leq \eta \|x^k - \tilde{x}^k\|, \qquad 0 < \eta < 1. \tag{14}$$

Correction step: For $\alpha > 0$ and $0 \leq a_1 < 1$, the new iterate $x^{k+1}(\alpha)$ is defined by

$$x^{k+1}(\alpha) = a_1 x^k + (1 - a_1) P_{R_+^n} \left[x^k - \frac{\alpha \beta_k}{1+\mu} F(\tilde{x}^k) \right]. \tag{15}$$

Remark 3.1 (14) implies that

$$|(x^k - \tilde{x}^k)^T \xi^k| \leq \eta \|x^k - \tilde{x}^k\|^2, \quad 0 < \eta < 1. \tag{16}$$

Remark 3.2 In [1], Auslender et al. proposed the following conditions

$$\sum_{k=1}^{\infty} \|\xi^k\| < +\infty \quad \text{and} \quad \sum_{k=1}^{\infty} \langle \xi^k, x^k \rangle < +\infty, \tag{17}$$

to ensure convergence. It seems that the accuracy criterion (14) can be checked and complemented in practice more easily than (17).

How to choose a suitable step length $\alpha > 0$ to force convergence? To answer this question, we consider maximizing the progress $\Theta(\alpha)$ (will be defined in (18)). Since the solution point x^* is unknown, we can not maximize $\Theta(\alpha)$ directly. The following Lemma 3.1 and Theorem 3.1 convert the task to maximize function $\Phi(\alpha)$ (will be defined in (30)) which does not contain x^*.

Lemma 3.1 *Let $x^* \in \Omega^*$, $x^{k+1}(\alpha)$ be defined by (15) and*

$$\Theta(\alpha) = \|x^k - x^*\|^2 - \|x^{k+1}(\alpha) - x^*\|^2, \tag{18}$$

then we have

$$\Theta(\alpha) \geq (1 - a_1)\{\|x^k - x_*^k\|^2 + 2\alpha(x_*^k - \tilde{x}^k)^T d^k - \frac{2\alpha\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2\}, \tag{19}$$

where

$$x_*^k := P_{R_+^n}\left[x^k - \frac{\alpha\beta_k}{1+\mu}F(\tilde{x}^k)\right] \quad \textit{and} \quad d^k := (x^k - \tilde{x}^k) + \frac{1}{1+\mu}\xi^k. \tag{20}$$

Proof By setting $q = \beta_k F(\tilde{x}^k) - \xi^k$ in (10) and $y = x_*^k := P_{R_+^n}\left[x^k - \frac{\alpha\beta_k}{1+\mu}F(\tilde{x}^k)\right]$ in (11), it follows

$$(x_*^k - \tilde{x}^k)^T \left(\frac{1}{1+\mu}(\xi^k - \beta_k F(\tilde{x}^k))\right) \leq \frac{1}{2}\left(\|x^k - x_*^k\|^2 - \|\tilde{x}^k - x_*^k\|^2\right) - \frac{1-\mu}{2(1+\mu)}\|x^k - \tilde{x}^k\|^2. \tag{21}$$

Using the following identity

$$(x_*^k - \tilde{x}^k)^T(x^k - \tilde{x}^k) = \tfrac{1}{2}\left(\|\tilde{x}^k - x_*^k\|^2 - \|x^k - x_*^k\|^2\right) + \tfrac{1}{2}\|x^k - \tilde{x}^k\|^2. \tag{22}$$

Adding (21) and (22) we then obtain

$$(x_*^k - \tilde{x}^k)^T\{(x^k - \tilde{x}^k) + \frac{1}{1+\mu}(\xi^k - \beta_k F(\tilde{x}^k))\} \leq \frac{\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2,$$

which implies

$$2\alpha(x_*^k - \tilde{x}^k)^T\{(x^k - \tilde{x}^k) + \frac{1}{1+\mu}(\xi^k - \beta_k F(\tilde{x}^k))\} - \frac{2\alpha\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2 \leq 0. \tag{23}$$

Since $x^* \in \Omega^* \subset R_+^n$ and $x_*^k = P_{R_+^n}\left[x^k - \frac{\alpha\beta_k}{1+\mu}F(\tilde{x}^k)\right]$, it follows from (9) that

$$\|x_*^k - x^*\|^2 \leq \|x^k - \frac{\alpha\beta_k}{1+\mu}F(\tilde{x}^k) - x^*\|^2 - \|x^k - \frac{\alpha\beta_k}{1+\mu}F(\tilde{x}^k) - x_*^k\|^2. \tag{24}$$

From (15), we get

$$\begin{aligned}\|x^{k+1}(\alpha)-x^*\|^2 &= \|a_1(x^k-x^*)+(1-a_1)(x_*^k-x^*)\|^2\\ &= a_1^2\|x^k-x^*\|^2+(1-a_1)^2\|x_*^k-x^*\|^2\\ &\quad +2a_1(1-a_1)(x^k-x^*)^T(x_*^k-x^*).\end{aligned}$$

Using the following identity

$$2(a+b)^Tb=\|a+b\|^2-\|a\|^2+\|b\|^2$$

for $a=x^k-x_*^k$, $b=x_*^k-x^*$ and (24), we obtain

$$\begin{aligned}\|x^{k+1}(\alpha)-x^*\|^2 &= a_1^2\|x^k-x^*\|^2+(1-a_1)^2\|x_*^k-x^*\|^2+a_1(1-a_1)\\ &\quad\times\{\|x^k-x^*\|^2-\|x^k-x_*^k\|^2+\|x_*^k-x^*\|^2\}\\ &= a_1\|x^k-x^*\|^2+(1-a_1)\|x_*^k-x^*\|^2-a_1(1-a_1)\|x^k-x_*^k\|^2\\ &\le a_1\|x^k-x^*\|^2+(1-a_1)\|x^k-\frac{\alpha\beta_k}{1+\mu}F(\tilde{x}^k)-x^*\|^2\\ &\quad -(1-a_1)\|x^k-\frac{\alpha\beta_k}{1+\mu}F(\tilde{x}^k)-x_*^k\|^2-a_1(1-a_1)\|x^k-x_*^k\|^2.\end{aligned} \tag{25}$$

Using the definition of $\Theta(\alpha)$ and (25), we get

$$\begin{aligned}\Theta(\alpha) &\ge (1-a_1^2)\|x^k-x_*^k\|^2+\frac{2(1-a_1)\alpha\beta_k}{1+\mu}(x_*^k-x^k)^TF(\tilde{x}^k)\\ &\quad +\frac{2(1-a_1)\alpha\beta_k}{1+\mu}(x^k-x^*)^TF(\tilde{x}^k).\end{aligned} \tag{26}$$

Since $\tilde{x}^k\in R_+^n$ and x^* is solution of VI(F, R_+^n), using pseudomonotonicity of F we have

$$(\tilde{x}^k-x^*)^TF(x^*)\ge 0\Rightarrow(\tilde{x}^k-x^*)^TF(\tilde{x}^k)\ge 0,$$

and consequently

$$(x^k-x^*)^TF(\tilde{x}^k)\ge(x^k-\tilde{x}^k)^TF(\tilde{x}^k). \tag{27}$$

Applying (27) to the last term in the right side of (26) and using $0\le a_1<1$, we obtain

$$\begin{aligned}\Theta(\alpha) &\ge (1-a_1^2)\|x^k-x_*^k\|^2+\frac{2(1-a_1)\alpha\beta_k}{1+\mu}(x_*^k-\tilde{x}^k)^TF(\tilde{x}^k)\\ &\ge (1-a_1)\{\|x^k-x_*^k\|^2+\frac{2\alpha\beta_k}{1+\mu}(x_*^k-\tilde{x}^k)^TF(\tilde{x}^k)\}.\end{aligned} \tag{28}$$

Adding (23) (multiplied by $1-a_1$) to (28) and using the notation of d^k in (20), the Lemma is proved. □

Theorem 3.1 *Let $\Theta(\alpha)$ be defined in (18) and d^k be defined in (20), then for any $x^*\in\Omega^*$ and $\alpha>0$, we have*

$$\Theta(\alpha)\ge(1-a_1)\Phi(\alpha), \tag{29}$$

where

$$\Phi(\alpha)=2\alpha\varphi_k-\alpha^2\|d^k\|^2, \tag{30}$$

and

$$\varphi_k = \frac{1}{1+\mu}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T \xi^k. \tag{31}$$

Proof It follows from (19) that

$$\begin{aligned}
\Theta(\alpha) &\geq (1-a_1)\{\|x^k - x_*^k\|^2 + 2\alpha(x_*^k - x^k)^T d^k + 2\alpha(x^k - \tilde{x}^k)^T d^k - \frac{2\alpha\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2\} \\
&= (1-a_1)\{\|x_*^k - x^k + \alpha d^k\|^2 - \alpha^2\|d^k\|^2 + 2\alpha(x^k - \tilde{x}^k)^T d^k - \frac{2\alpha\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2\} \\
&\geq (1-a_1)\{2\alpha(x^k - \tilde{x}^k)^T(x^k - \tilde{x}^k + \frac{1}{1+\mu}\xi^k) - \frac{2\alpha\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2 - \alpha^2\|d^k\|^2\} \\
&= (1-a_1)\left(2\alpha\{\frac{1}{1+\mu}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T\xi^k\} - \alpha^2\|d^k\|^2\right) \\
&= (1-a_1)\{2\alpha\varphi_k - \alpha^2\|d^k\|^2\},
\end{aligned}$$

and (29) is proved. □

4. Convergence analysis

$\Phi(\alpha)$ measures the progress obtained in the kth iteration. It is natural to choose a step length α_k which maximizes the progress. Note that $\Phi(\alpha)$ is a quadratic function of α and it reaches its maximum at

$$\alpha_k^* = \frac{\varphi_k}{\|d^k\|^2}, \tag{32}$$

and

$$\Phi(\alpha_k^*) = \alpha_k^* \varphi_k. \tag{33}$$

In the next theorem we show that α_k^* and $\Phi(\alpha_k^*)$ are lower bounded away from zero, and it is one of the keys to prove the global convergence results.

Theorem 4.1 *For given $x^k \in R_+^n$ and $\beta_k > 0$, let $\tilde{x}^k$ and ξ^k satisfied to the condition (14), then we have the following,*

$$\alpha_k^* \geq \frac{1-\eta}{2(1+\mu)}, \tag{34}$$

and

$$\Phi(\alpha_k^*) \geq \frac{(1-\eta)^2}{2(1+\mu)^2}\|x^k - \tilde{x}^k\|^2. \tag{35}$$

Proof It follows from (31) to (16) that

$$\varphi_k = \frac{1}{1+\mu}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T\xi^k \geq \left(\frac{1-\eta}{1+\mu}\right)\|x^k - \tilde{x}^k\|^2. \tag{36}$$

If $(x^k - \tilde{x}^k)^T \xi^k \le 0$, since $\mu > 0$ it follows from (14) to (20) that

$$\begin{aligned}\|d^k\|^2 &\le \|x^k - \tilde{x}^k\|^2 + \frac{1}{(1+\mu)^2}\|\xi^k\|^2 \\ &\le \|x^k - \tilde{x}^k\|^2 + \|\xi^k\|^2 \\ &\le 2\|x^k - \tilde{x}^k\|^2, \end{aligned} \tag{37}$$

from (36) to (37), we obtain

$$\alpha_k^* = \frac{\varphi_k}{\|d^k\|^2} \ge \frac{1-\eta}{2(1+\mu)}.$$

Otherwise, if $(x^k - \tilde{x}^k)^T \xi^k \ge 0$, it follows from $0 < \mu < 1$, $0 < \eta < 1$ and (14) that

$$\begin{aligned}\varphi_k &= \frac{1}{1+\mu}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T \xi^k \\ &\ge \frac{1}{1+\mu}\{\frac{1}{2}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T \xi^k \\ &\quad + \frac{1}{2}\|x^k - \tilde{x}^k\|^2\} \\ &\ge \frac{1}{1+\mu}\{\frac{1}{2}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T \xi^k \\ &\quad + \frac{1}{2(1+\mu)^2}\|\xi^k\|^2\} \\ &= \frac{1}{2(1+\mu)}\|d^k\|^2, \end{aligned}$$

and thus

$$\alpha_k^* \ge \frac{1}{2(1+\mu)} \ge \frac{1-\eta}{2(1+\mu)}.$$

Using (33), (34) and (36) directly we obtained (35). □

For fast convergence, we take a relaxation factor $\gamma \in [1, 2)$ and set the step-size α_k in (15) by $\alpha_k = \gamma \alpha_k^*$. Through simple manipulations we obtain

$$\begin{aligned}\Phi(\gamma\alpha_k^*) &= 2\gamma\alpha_k^*\varphi_k - (\gamma^2\alpha_k^*)(\alpha_k^*\|d^k\|^2) \\ &= (2\gamma\alpha_k^* - \gamma^2\alpha_k^*)\varphi_k \\ &= \gamma(2-\gamma)\Phi(\alpha_k^*). \end{aligned} \tag{38}$$

It follows from Theorem 3.1 and Theorem 4.1 that there is a constant

$$c := \frac{\gamma(2-\gamma)(1-a_1)(1-\eta)^2}{2(1+\mu)^2} > 0,$$

such that

$$\|x^{k+1} - x^*\|^2 \le \|x^k - x^*\|^2 - c\|x^k - \tilde{x}^k\|^2 \qquad \forall x^* \in \Omega^*. \tag{39}$$

The following result is very useful to prove the convergence of our method.

Lemma 4.2 *For given $x^k > 0$ and $\beta_k > 0$, let $\tilde{x}^k$ to be obtained by the prediction step* (13), *then for each $x \ge 0$ we have*

$$(x - \tilde{x}^k)^T(\beta_k F(\tilde{x}^k) - \xi^k) \ge (x^k - \tilde{x}^k)^T\{(1+\mu)x - (\mu x^k + \tilde{x}^k)\}. \tag{40}$$

Proof By setting $q = \beta_k F(\tilde{x}^k) - \xi^k$ in (10) and $y = x$ in (11), it follows from (13) that

$$\begin{aligned}(x-\tilde{x}^k)^T(\beta_k F(\tilde{x}^k)-\xi^k) \geq\ & \frac{1+\mu}{2}(\|\tilde{x}^k - x\|^2 - \|x^k - x\|^2) + \frac{1-\mu}{2}\|x^k - \tilde{x}^k\|^2 \\ =\ & (1+\mu)x^T x^k - (1+\mu)x^T\tilde{x}^k - (1-\mu)(\tilde{x}^k)^T x^k \\ & -\mu\|x^k\|^2 + \|\tilde{x}^k\|^2 \\ =\ & (1+\mu)x^T(x^k - \tilde{x}^k) - (x^k - \tilde{x}^k)^T(\mu x^k + \tilde{x}^k) \\ =\ & (x^k - \tilde{x}^k)^T\{(1+\mu)x - (\mu x^k + \tilde{x}^k)\}.\end{aligned}$$

Then the proof is completed. □

Now, the convergence of the proposed method could be proved as follows:

Theorem 4.2 *If* $\inf_{k=0}^{\infty} \beta_k = \beta > 0$, *then the sequence* $\{x^k\}$ *generated by the proposed method converges to some* x^{∞} *which is a solution of* $VI(F, R^n_+)$.

Proof It follows from (39) that $\{x^k\}$ is a bounded sequence and

$$\lim_{k\to\infty} \|x^k - \tilde{x}^k\| = 0. \tag{41}$$

Consequently, $\{\tilde{x}^k\}$ is also bounded. Since $\lim_{k\to\infty}\|x^k - \tilde{x}^k\| = 0,\ \|\xi^k\| \leq \eta\|x^k - \tilde{x}^k\|$ and $\beta_k \geq \beta > 0$, it follows from (40) that

$$\lim_{k\to\infty}(x - \tilde{x}^k)^T F(\tilde{x}^k) \geq 0, \quad \forall x \in R^n_+.$$

Because $\{\tilde{x}^k\}$ is bounded, it has at least one cluster point. Let x^{∞} be a cluster point of $\{\tilde{x}^k\}$ and the subsequence $\{\tilde{x}^{k_j}\}$ converges to x^{∞}. It follows that

$$\lim_{j\to\infty}(x - \tilde{x}^{k_j})^T F(\tilde{x}^{k_j}) \geq 0, \quad \forall x \in R^n_+,$$

and consequently

$$(x - x^{\infty})^T F(x^{\infty}) \geq 0, \quad \forall x \in R^n_+.$$

Then x^{∞} is a solution of VI(F, R^n_+). Note that the inequality (39) is true for all solution points of VI(F, R^n_+) and hence we have

$$\|x^{k+1} - x^{\infty}\|^2 \leq \|x^k - x^{\infty}\|^2, \quad \forall k \geq 0. \tag{42}$$

Since $\tilde{x}^{k_j} \to x^{\infty}(j \to \infty)$ and $x^k - \tilde{x}^k \to 0(k \to \infty)$, for any $\epsilon > 0$, there exists an $l > 0$ such that

$$\|\tilde{x}^{k_l} - x^{\infty}\| < \frac{\epsilon}{2} \quad \text{and} \quad \|x^{k_l} - \tilde{x}^{k_l}\| < \frac{\epsilon}{2}. \tag{43}$$

Therefore, for any $k \geq k_l$, it follows from (42) to (43) that

$$\|x^k - x^{\infty}\| \leq \|x^{k_l} - x^{\infty}\| \leq \|x^{k_l} - \tilde{x}^{k_l}\| + \|\tilde{x}^{k_l} - x^{\infty}\| < \epsilon.$$

This implies that the sequence $\{x^k\}$ converges to x^{∞} which is a solution of VI(F, R^n_+). □

5. Preliminary computational results

For numerical experiment we need to find the values of the approximate solution $\tilde{x}^k$. In the special case $\xi^k = \beta_k(F(\tilde{x}^k) - F(x^k))$, then (13) is equivalent to the following system of nonlinear equations

$$\beta_k F(x^k) + \tilde{x}^k - x^k + \mu X_k \log \frac{\tilde{x}^k}{x^k} = 0, \tag{44}$$

hence

$$\tilde{x}^k_j + \mu x^k_j \log \tilde{x}^k_j + (\beta_k F_j(x^k) - x^k_j - \mu x^k_j \log x^k_j) = 0, \quad j = 1, \ldots, n. \tag{45}$$

The recursion of classical Newton method for the above problem is

$$\tilde{x_j}^k := x_j{}^k - \frac{\beta_k}{1+\mu} F_j(x^k).$$

The solution of (45) is $\tilde{x}^k > 0$, to avoid the non-positive value $\tilde{x_j}^k$ in the iteration process, we take

$$\tilde{x_j}^k := \max\{x_j{}^k - \frac{\beta_k}{1+\mu} F_j(x^k), 0\}, \quad j = 1, \ldots, n.$$

The detailed algorithm is as follows.
Step 0. Let $\beta_0 = 1, \eta (:= 0.95) < 1, 0 \le a_1 < 1, \mu = 0.1, \gamma = 1.9, \epsilon = 10^{-7}, k = 0$ and $x^0 \in R^n_+$.
Step 1. If $\| \min(x, F(x)) \|_\infty \le \epsilon$, then stop. Otherwise, go to Step 2.
Step 2. (Prediction step)

$$\tilde{x}^k = P_{R^n_+}[x^k - \tfrac{\beta_k}{1+\mu} F(x^k)], \qquad \xi^k := \beta_k(F(\tilde{x}^k) - F(x^k)),$$

$r := \|\xi^k\| / \|x^k - \tilde{x}^k\|$.
while $(r > \eta)$

$$\beta_k := \beta_k * 0.8/r, \qquad \tilde{x}^k = P_{R^n_+}[x^k - \tfrac{\beta_k}{1+\mu} F(x^k)],$$

$$\xi^k := \beta_k(F(\tilde{x}^k) - F(x^k)), \qquad r := \|\xi^k\| / \|x^k - \tilde{x}^k\|.$$

end while

Step 3. (correction step)

$$\varphi_k = \tfrac{1}{1+\mu} \|x^k - \tilde{x}^k\|^2 + \tfrac{1}{1+\mu} (x^k - \tilde{x}^k) \xi^k,$$

$$d^k = (x^k - \tilde{x}^k) + \tfrac{1}{1+\mu} \xi^k, \qquad \alpha_k = \gamma \alpha_k^* = \gamma \tfrac{\varphi_k}{\|d^k\|^2},$$

$$x^{k+1} = a_1 x^k + (1 - a_1) P_{R^n_+}[x^k - \tfrac{\alpha_k \beta_k}{1+\mu} F(\tilde{x}^k)],$$

Step 4. $\beta_{k+1} = \begin{cases} \frac{\beta_k * 0.7}{r}, & \text{if } r \le 0.5; \\ \beta_k, & \text{otherwise.} \end{cases}$

Step 5. k:=k+1; go to Step 1.

5.1. Comparison with other methods

(1) A well known projection method is the extragradient method of Korpelevich [11] which was extended by Khobotov [10]. The proposed method also can be viewed as improvement of [10] in two directions:
Firstly, instead of $\alpha_k \equiv 1$ we have proposed $\alpha_k = \gamma\alpha_k^*$ is dependent on the current point x^k, $\tilde{x}^k$ and ξ^k, and thus more precise.
Secondly the parameter sequence $\{\beta_k\}$ in the Khobotov method is monotonically non increasing. However, this may cause a slow convergence if β_k is taken too small. To overcome this difficulty, we have proposed a self-adaptive technique. The main contribution of this technique is that we allow elements of the penalty sequence to either increase or decrease in iterations, not necessarily monotone.

(2) For given x^k and $\beta_k := \frac{\beta_k}{1+\mu} > 0$, denote $F_k(x) := (x - x^k) + \beta_k F(x)$ and $\Omega = R^n_+$. The iteration of Solodov and Svaiter's method (see [14] pp. 385, Algorithm 2) consists of the following steps:
Algorithm SS
Step 1. Find a y^k which is an approximate solution of

$$x \in R^n_+, \quad (x' - x)^T F_k(x) \geq 0, \quad \forall x' \in R^n_+, \tag{46}$$

such that

$$\{y^k - P_{R^n_+}[y^k - F_k(y^k)]\}^T F_k(y^k) - \frac{1}{2}\|y^k - P_{R^n_+}[y^k - F_k(y^k)]\|^2 \leq \frac{\eta}{2}\|y^k - x^k\|^2. \tag{47}$$

Step 2. Set

$$x^{k+1} = P_{R^n_+}[x^k - \beta_k F(y^k)]. \tag{48}$$

Note that the term y^k in Algorithm SS plays the same role as the term $\tilde{x}^k$ in our method. Now let us observe the differences between Algorithm SS and our framework.
First we compare the error restrictions of the two methods. In Step 1 of Algorithm SS, since $y^k \in R^n_+$, it follows from (7) that

$$\{y^k - P_{R^n_+}[y^k - F_k(y^k)]\}^T F_k(y^k) \geq \|y^k - P_{R^n_+}[y^k - F_k(y^k)]\|^2.$$

In order to satisfy Condition (47), one needs at least

$$\{y^k - P_{R^n_+}[y^k - F_k(y^k)]\}^T F_k(y^k) \leq \eta\|y^k - x^k\|^2. \tag{49}$$

As $x^k \to x^*$, the direction $\left(y^k - P_{R^n_+}[y^k - F_k(y^k)]\right)$ is almost parallel to $F_k(y^k)$, and usually $F(x^*) \neq 0$. Therefore, as $x^k \to x^*$, it follows from (47) that

$$\|y^k - P_{R^n_+}[y^k - F_k(y^k)]\| = O(\|y^k - x^k\|^2). \tag{50}$$

Notice that the $\tilde{x}^k$ generated from our method (see Step 2) can be written as

$$\tilde{x}^k = P_{R^n_+}[\tilde{x}^k - F_k(\tilde{x}^k) + \varepsilon^k], \tag{51}$$

and it requires at most

$$\|\varepsilon^k\| \leq \eta \|\tilde{x}^k - x^k\|.$$

It is worthy to discuss the relation between ε^k and $e^k := y^k - P_{R^n_+}[y^k - F_k(y^k)]$ in formula (49). Note that $y^k = \tilde{x}^k$. Hence according to (51) we have

$$e^k = P_{R^n_+}[\tilde{x}^k - F_k(\tilde{x}^k) + \varepsilon^k] - P_{R^n_+}[y^k - F_k(y^k)].$$

Since the projection is nonexpansive, in general we have $\|e^k\| \leq \|\varepsilon^k\|$. Therefore compared with Algorithm SS, the proposed method has a much relaxed error restriction.

Next we compare the step lengths employed in the correction step. In Algorithm SS, the step length is $\alpha_k \equiv 1$, which is different from the step length in our framework.

To test the proposed method, we consider the nonlinear complementarity problems:

$$x \geq 0 \quad F(x) \geq 0, \quad x^T F(x) = 0, \tag{52}$$

where

$$F(x) = D(x) + Mx + q,$$

$D(x)$ and $Mx + q$ are the nonlinear part and linear parts of $F(x)$, respectively.

The matrix $M = A^T A + B$ is computed as follows. A is $n \times n$ matrix whose entries are randomly generalized in the interval $(-5, +5)$ and the skew-symmetric matrix B is generated in the same way. The vector q is generated from a uniform distribution in the interval $(-200, 300)$. The components of D(x) are $D_j(x) = d_j * arctan(x_j)$ and d_j is chosen randomly in $(0, 1)$.

In all tests we take the logarithmic proximal parameter $\mu = 0.01, a_1 = 0.01$ and all iterations start with $x^0 = (0, \ldots, 0)^T$ or $x^0 = (1, \ldots, 1)^T$. We also test these problems with $\alpha = 1$, we denote by LQP$_1$ and LQP$_\alpha$ as the proposing methods with $\alpha = 1$ and $\alpha = \gamma\alpha^*$, respectively. The stop criterion was set to be

$$\| \min(x^k, F(x^k))\|_\infty \leq 10^{-7}.$$

All codes were written in Matlab, since the random value in Matlab is time dependent, we tested each problem five times. The iteration numbers and the computational time for the problem with different dimensions are given in the following tables.

Table 5.1 Numerical results for (52) problem with start point $x^0 = (0, \ldots, 0)^T$

Dimension of the problem	LQP$_1$ method		LQP$_\alpha$ method	
	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)
n=100	363	1.22	211	0.09
n=200	422	0.66	240	0.43
n=400	450	3.19	266	2.05
n=600	440	9.10	255	5.58
n=800	446	21.76	261	13.28

Table 5.2 Numerical results for (52) problem with start point $x^0 = (1, \ldots, 1)^T$

Dimension of the problem	LQP_1 method		LQP_α method	
	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)
n=100	407	0.14	219	0.11
n=200	451	0.72	251	0.52
n=400	489	3.46	279	2.37
n=600	514	10.72	293	6.87
n=800	546	26.32	308	16.49

Table 5.1 and Table 5.2 show that the proposed method is more efficient, the step size $\alpha = \gamma\alpha^*$ plays important role to reduce the iterative numbers due the fact that the step size $\alpha = \gamma\alpha^*$ is dependent on the current point x^k, $\tilde{x}^k$ and ξ^k, and thus more precise.

6. Concluding remarks

Based on the LQP method for VI(F, R^n_+), we suggested using a prediction–correction method to solve VI(F, R^n_+). The predictor is obtained via solving an inexact LQP method, with more relaxed accuracy criterion than (17) and the new iterate is computed by using the projection operator. Under suitable conditions, we proved the global convergence of the proposed method. The numerical results showed that our algorithm works well for the problem tested.

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